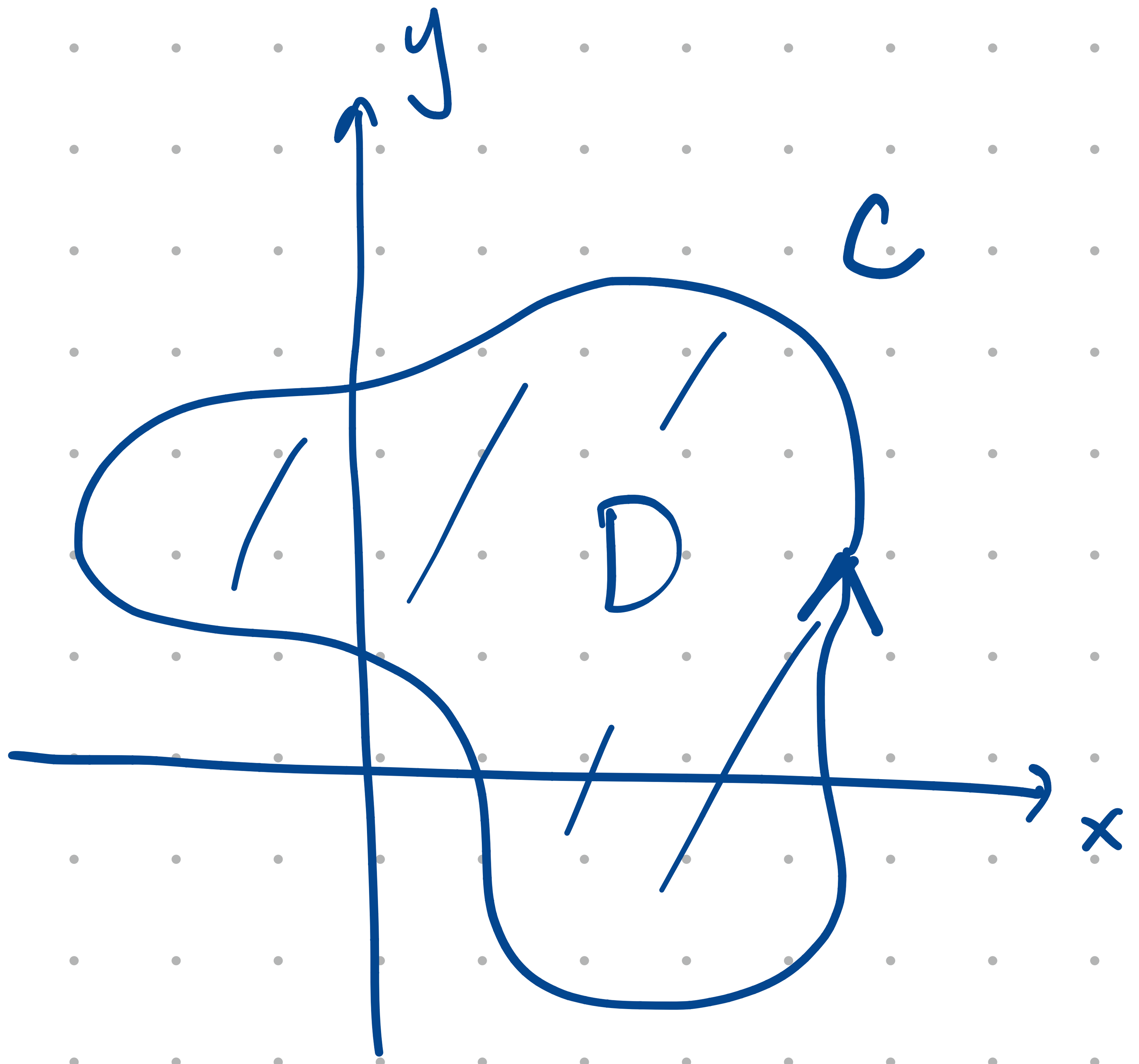
#1)



$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

$$\|$$

$$\iint_D (Q_x - P_y)\, dA$$

$$\vec{F} = \langle P, Q \rangle$$

To be conservative, $\oint_S \vec{F}\cdot d\vec{r} = 0$ for all closed loops S, which you are not given in the problem

Example. Let C be unit circle $x^2+y^2=1$ ccw

$$\vec{F} = \langle 0, x^2 \rangle$$

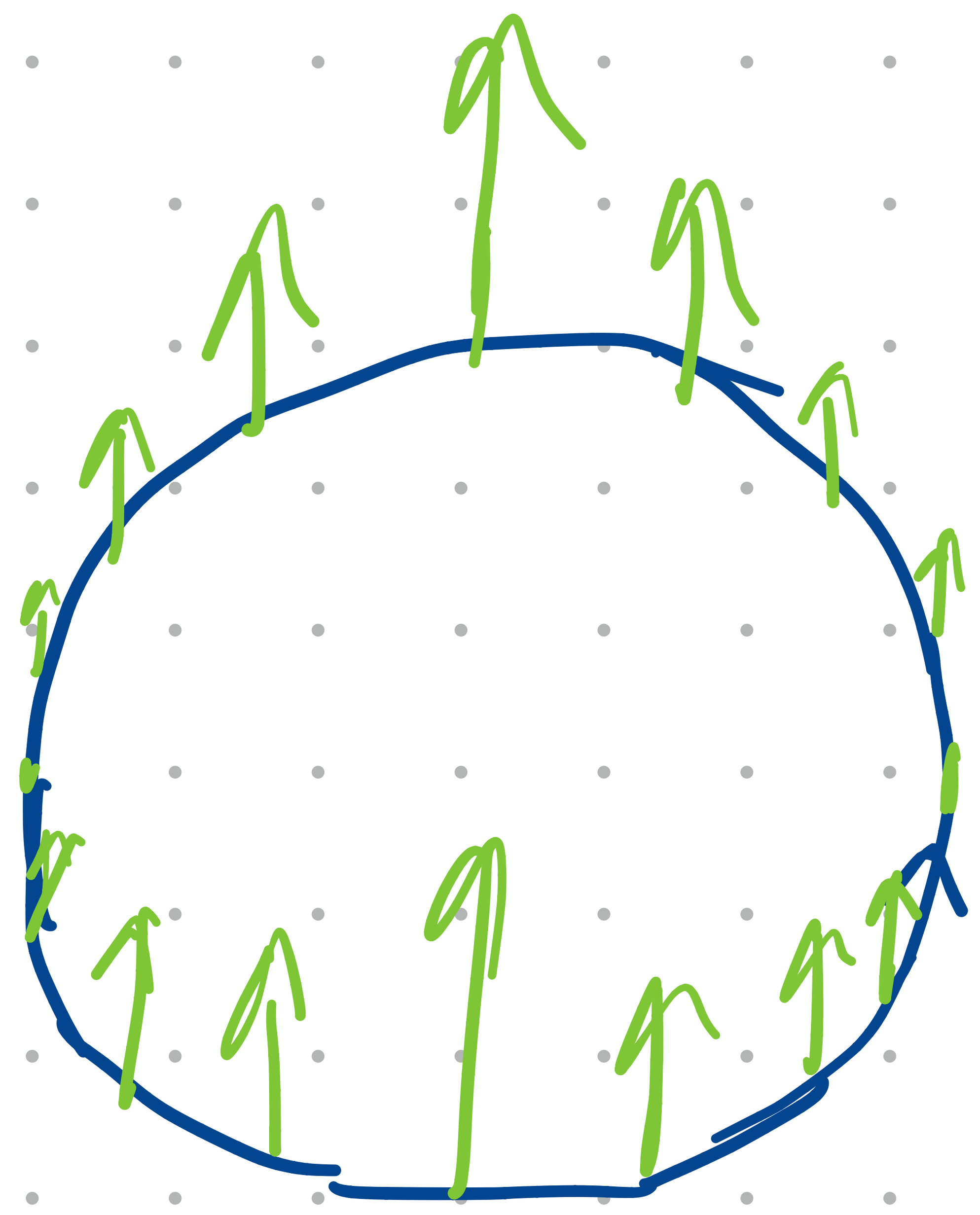
Then $$\oint_{\substack{x^2+y^2=1 \\ \text{ccw}}} \vec{F}\cdot d\vec{r} = \iint_{x^2+y^2\le 1} 2x \, dA$$

$$= \int_{-1}^{1} \boxed{\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2x \; dx} \; dy = 0$$

$=0$ b/c $2x$ is an odd fn. (of x)

<u>but</u> $Q_x - P_y = 2x$ which is not identically zero

and also



so $\vec{F}$ is not always perp. to C.

⚠ TL;DR: the three conditions in the poll <u>imply</u> that $\oint_C \vec{F} \cdot d\vec{r} = 0$, but are not implied by it for a particular closed loop C.

#2) $\oint_C \langle x,y\rangle \cdot d\vec{r} = 0$ for all curves C,

so this does not compute area enclosed.

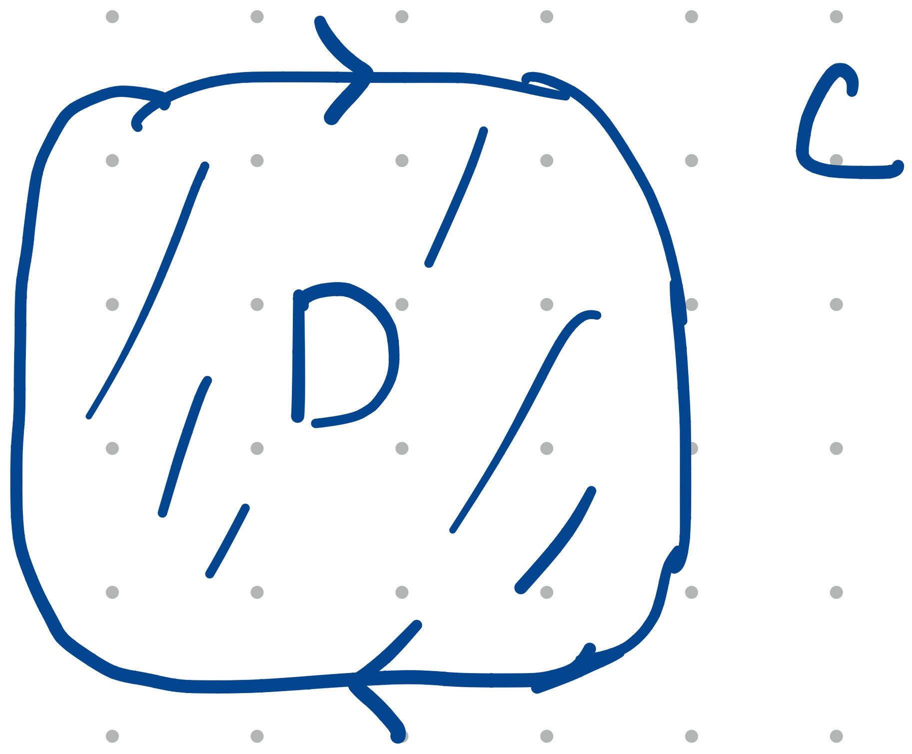
$$\oint_C \langle -y, x\rangle \cdot d\vec{r} = \iint_D 2\, dA = 2\cdot \text{Area}(D)$$

$$\oint_C \langle 5+x^2, x\rangle \cdot d\vec{r} = \iint_D 1\, dA = \text{Area}(D)$$

In practice, one would probably use $\langle 0, x\rangle$ or $\langle -y, 0\rangle$ etc.

Something simple w/ $Q_x - P_y = 1$.

#3)



$$\oint_C \underset{\uparrow}{x^3\,dy} = -\underbrace{\iint_D 3x^2\,dA}_{\text{positive quantity}} < 0.$$

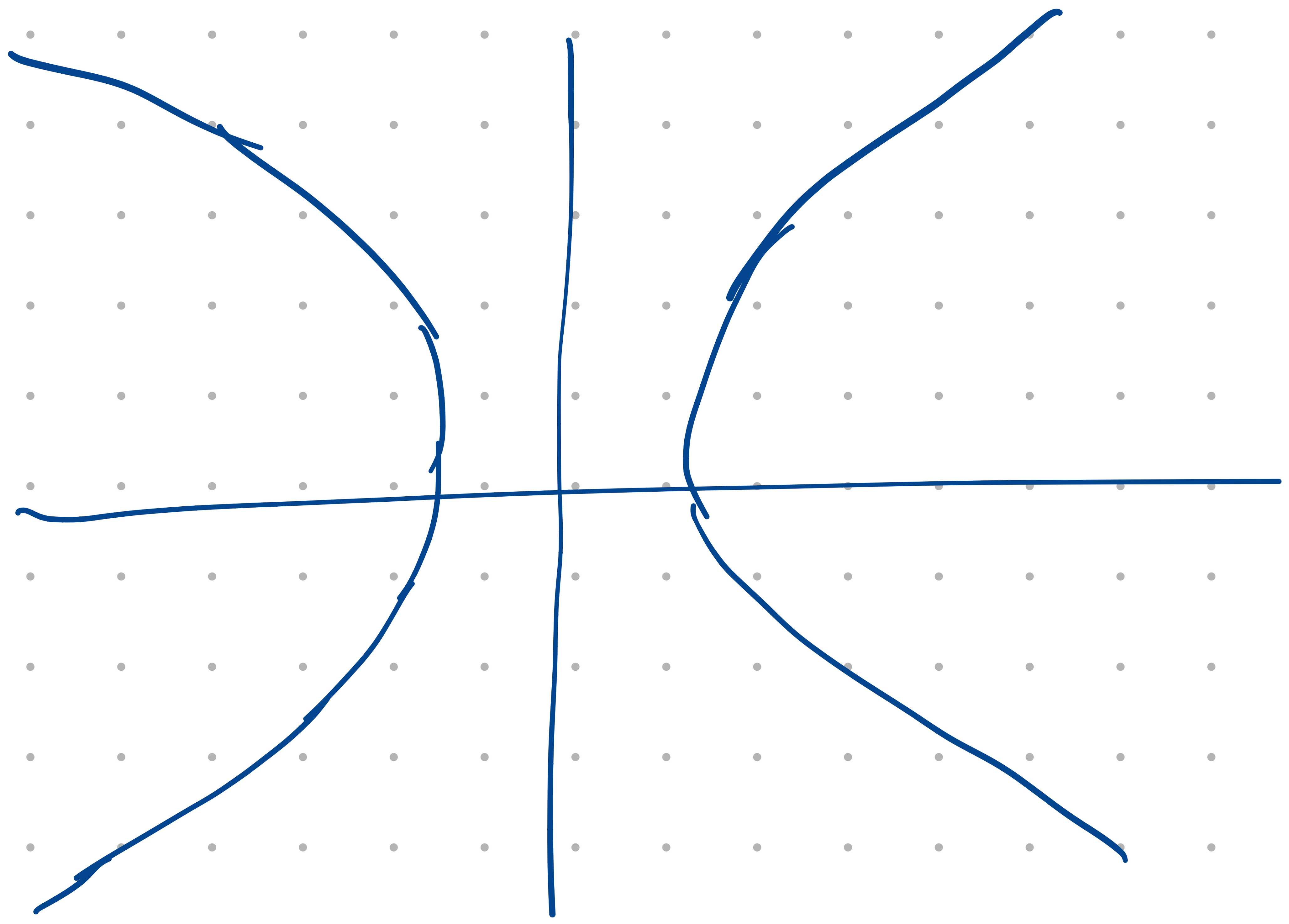
$\langle 0, x^3\rangle \cdot d\vec{r}$

#4) To interpret $z = f(x,y)$ as a level set

$$\underbrace{f(x,y) - z}_{F(x,y,z)} = 0$$

$$\nabla F(a,b,f(a,b)) = \langle f_x(a,b), f_y(a,b), -1\rangle$$

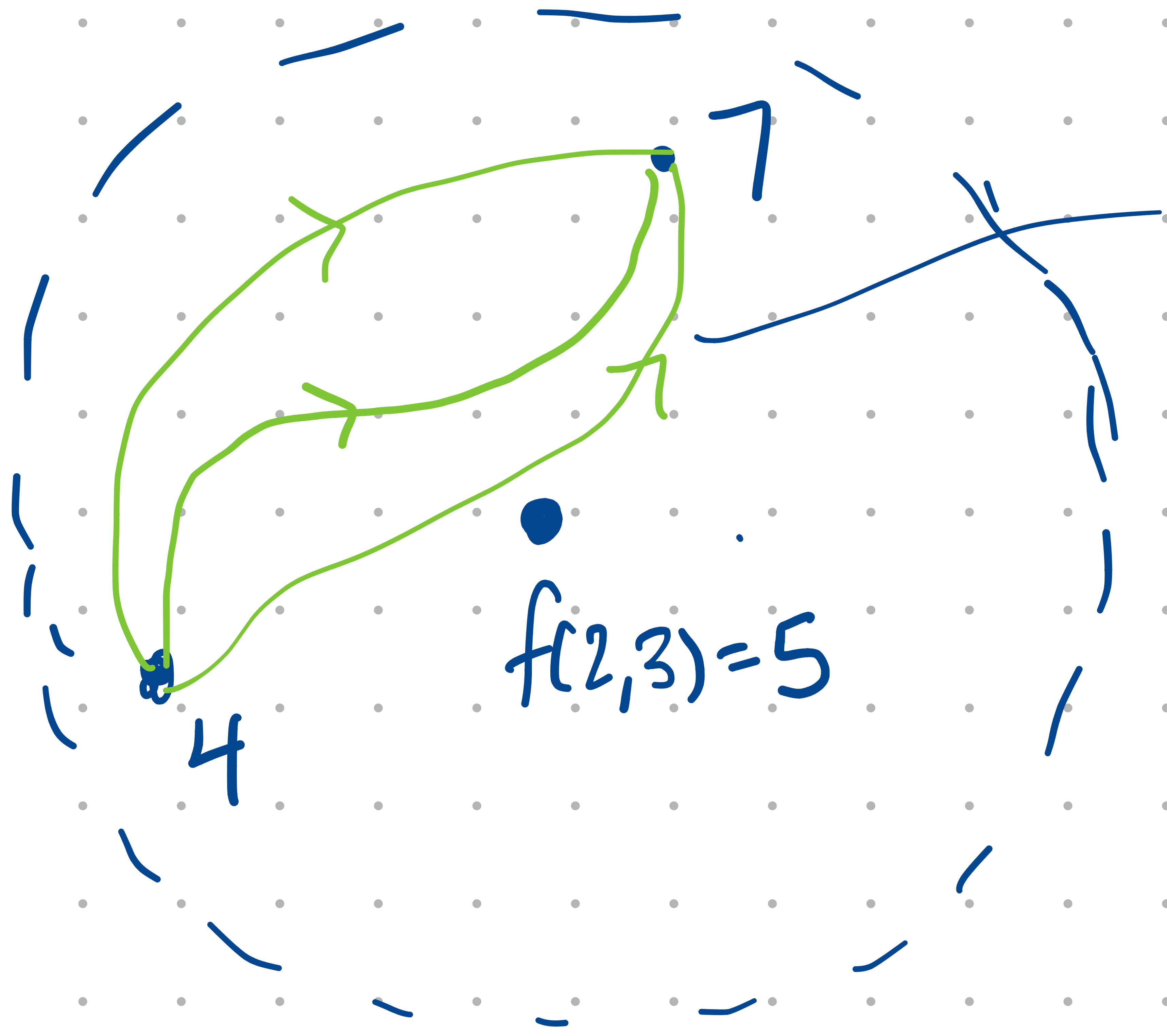
#5) Can't invoke EVT b/c $x^2-y^2=1$ is not bounded (although it is closed)



ex) $f(x,y)=y$ has no max or min on this curve.

#6) The 5-level set of $f(x,y)$ is a single point.

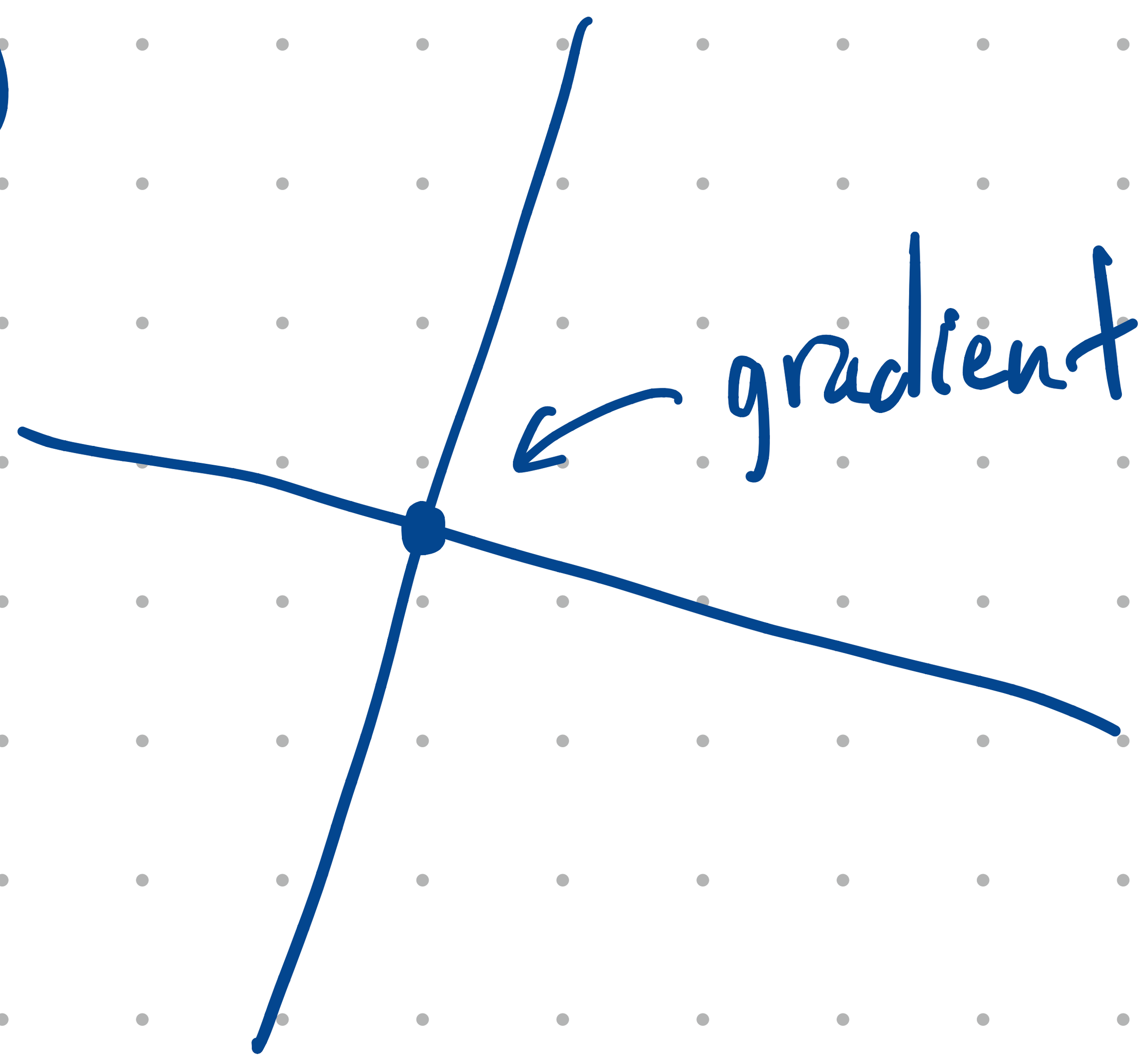
This can only happen if the rest of the graph $z=f(x,y)$ is either above or below $z=5$.



along each path, there must be a pt. where $f(x,y)=5$.

Conclusion: $(2,3)$ is either local min or max.

#7)



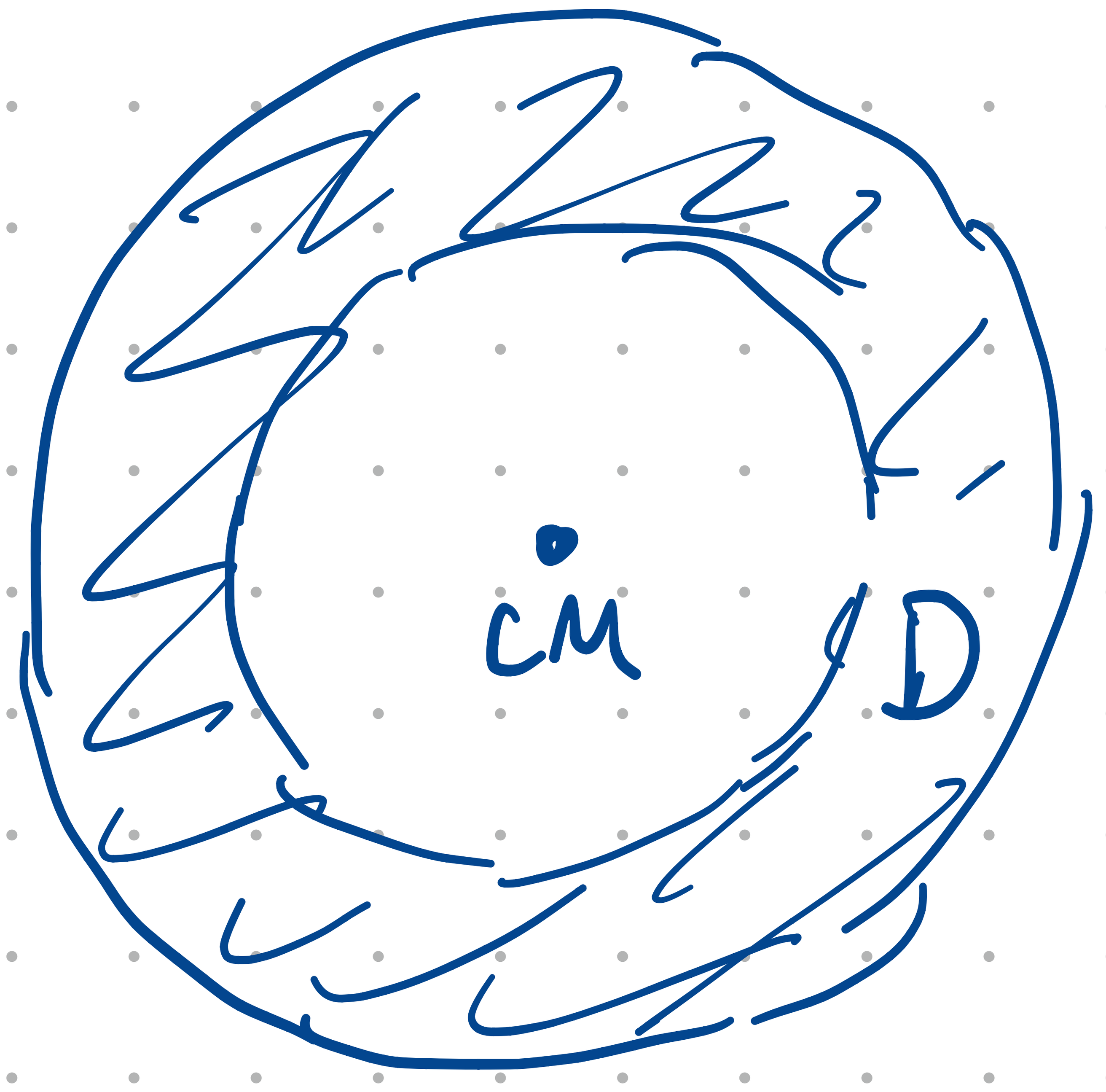
gradient cannot be nonzero here

$$f(x,y) = xy + 5 \implies \text{saddle}$$

$$f(x,y) = x^2y^2 + 5 \implies \text{local min}$$

etc.

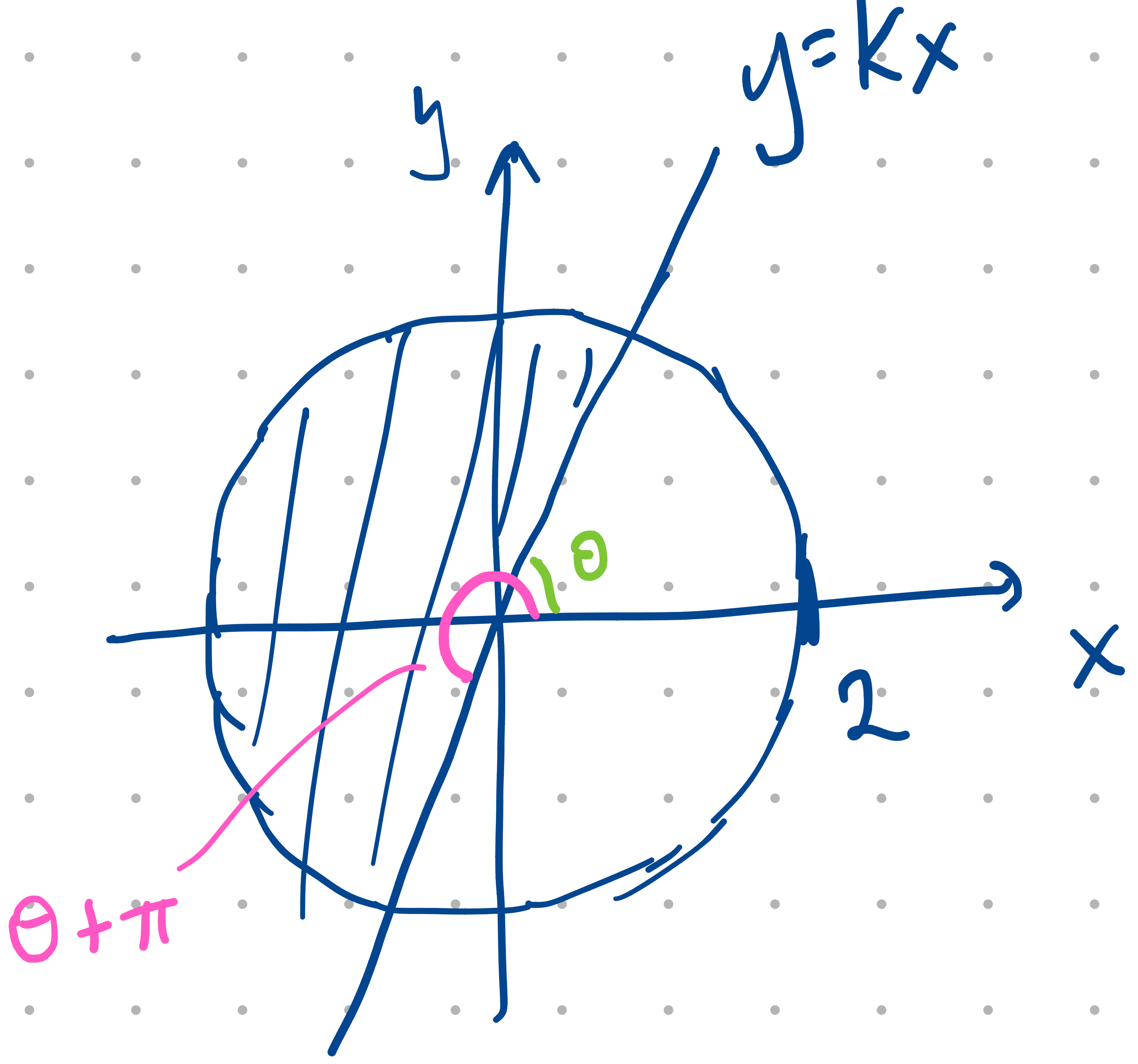
#8)



with uniform mass distr.

(Rmk: the CM must be in the "convex hull" of D, but that beyond the scope of what we discussed)

#9) No:



$$\int_\theta^{\theta+\pi}\int_0^2 e^{1+r^2}\, r\,dr\,d\theta$$

$$= \int_0^2 e^{1+r^2}\, r\,dr \int_\theta^{\theta+\pi} d\theta$$

$$= \pi \int_0^2 e^{1+r^2}\, r\,dr.$$

Ans is independent of $\theta = \arctan k$.

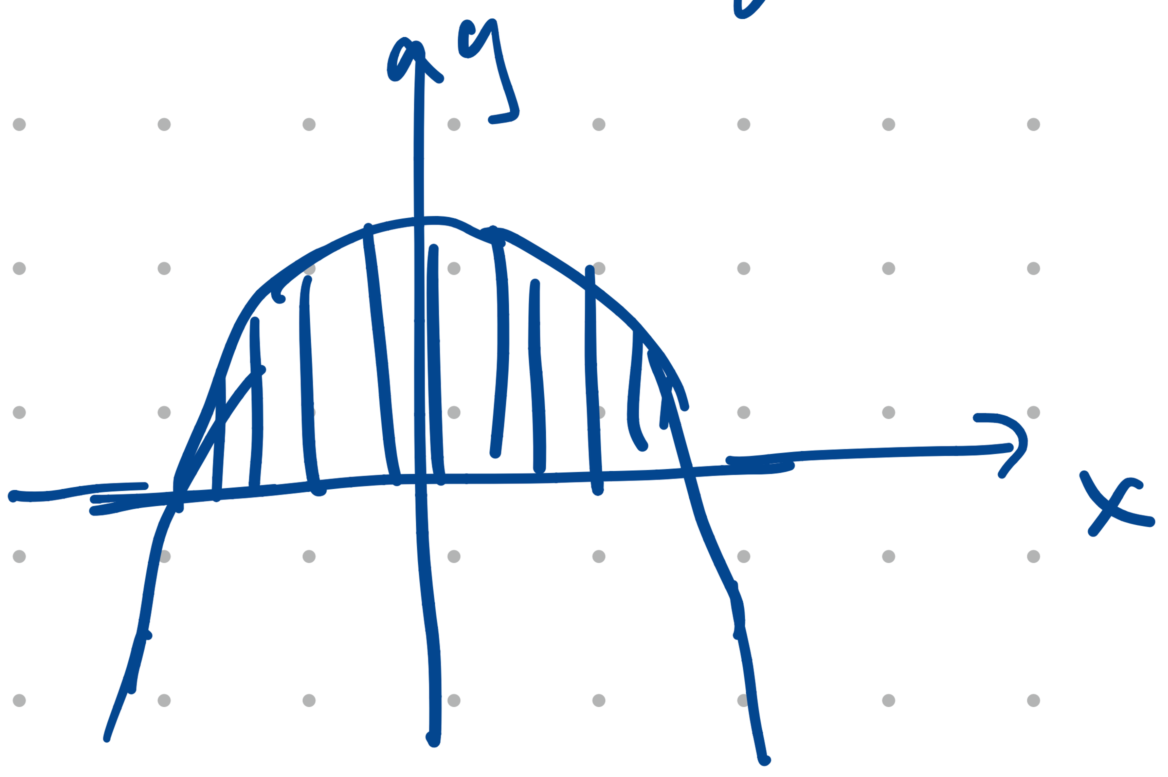
#10) Let D be the region enclosed by C.

Then $\oint_C (2y^3\,dx + (x-x^3)\,dy)$

$$= \iint_D (1-3x^2-6y^2)\,dA$$

We want to pick a region D which maximizes this integral.

<u>Example</u>: If you want to pick $a,b \in \mathbb{R}$ to maximize

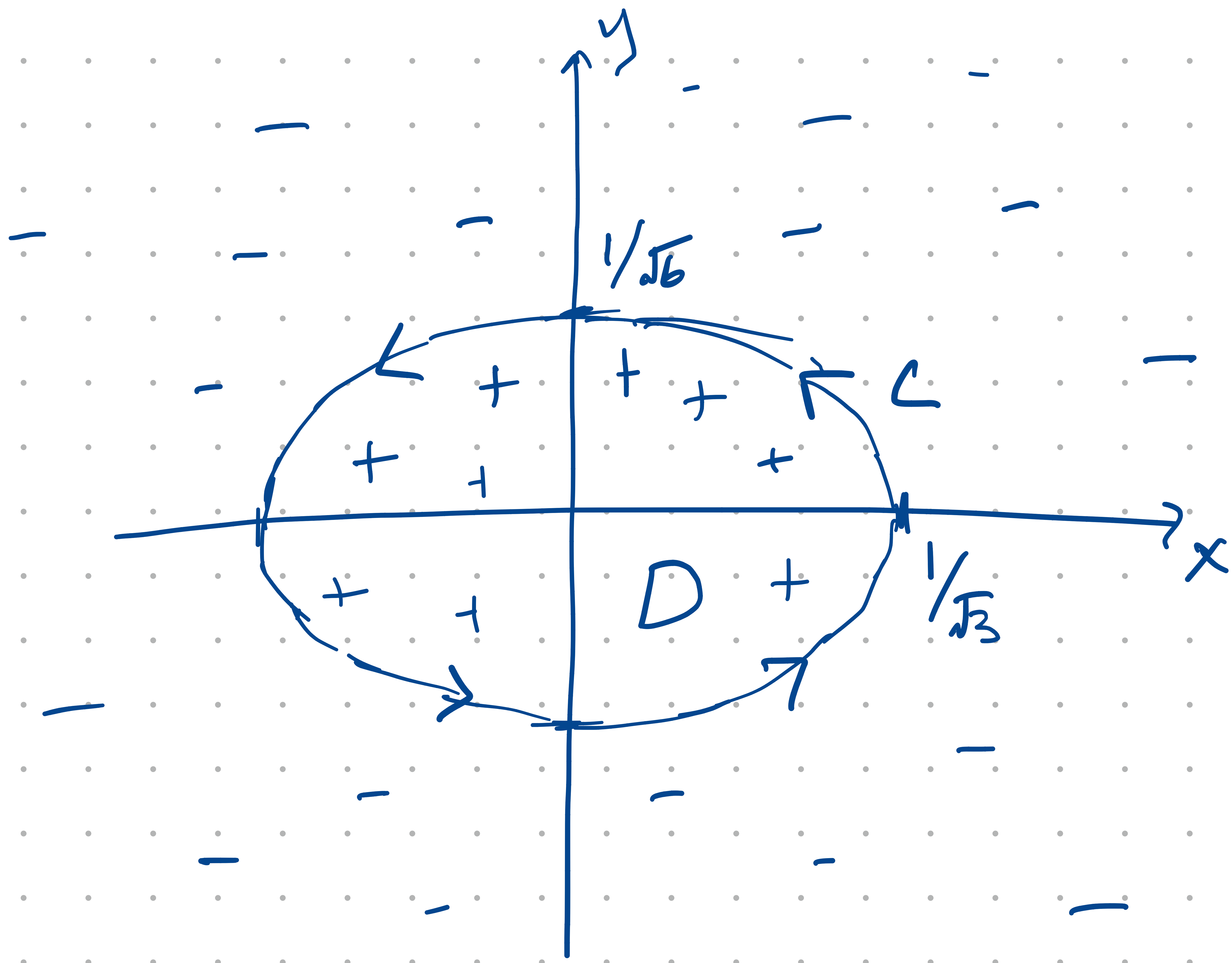


$$\int_a^b (1-x^2)\,dx,$$

should take $a=-1$, $b=1$

b/c $1-x^2 \geq 0$ from -1 to 1.

Same idea here: We take D to be $1-3x^2-6y^2 \geq 0$.



i.e. the double integral is maximized if we integrate over exactly the region on which the integrand is positive.

Now change vars: $x = \frac{u}{\sqrt{3}}$ $\quad y = \frac{v}{\sqrt{6}}$

So $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3\sqrt{2}}$

Then: $$\iint\limits_{1-3x^2-6y^2\geq 0} (1-3x^2-6y^2)\,dx\,dy$$

$$= \iint\limits_{1-u^2-v^2\geq 0} (1-u^2-v^2)\frac{1}{3\sqrt{2}}\,du\,dv$$

Change to polar: $u = r\cos\theta \quad v = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\frac{r}{3\sqrt{2}}\,dr\,d\theta$$

$$= \ldots\ldots = \boxed{\frac{\pi}{6\sqrt{2}}}$$